数理解析研究所講究録
第 1850 巻 2013 年 1-22

A geometric flow for quadrature surfaces

九州大学 マス・フォア・インダストリ研究所　小野寺 有紹

Michiaki Onodera
Institute of mathematics for industry,
Kyushu University

Abstract. A new geometric flow describing the motion of a closed surface is introduced. Moving surfaces evolving under the flow are shown to be a family of quadrature surfaces. It is proved that the geometric flow possesses a unique classical solution for any smooth initial surface with positive mean curvature.

1 Introduction

One of the classical problems in potential theory is to specify a closed surface Γ for a prescribed electric charge density μ in such a way that the uniform electric charge distribution on Γ induces the same potential in a neighborhood of the infinity as μ does. To formulate the problem mathematically, let F be the fundamental solution of $-\Delta$ in $\mathbb{R}^N$, i.e.,

$$F(x) := \begin{cases} -\dfrac{1}{2\pi}\log|x| & (N=2), \\ \dfrac{1}{N(N-2)\omega_N|x|^{N-2}} & (N\geq 3), \end{cases} \tag{1.1}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$, and let $\mathcal{H}^{N-1}\lfloor\Gamma$ denote the $(N-1)$-dimensional Hausdorff measure restricted to Γ. Then, the problem can be stated as follows: For a prescribed finite positive Radon measure μ with compact support in $\mathbb{R}^N$, find a $(N-1)$-dimensional closed surface Γ enclosing a bounded domain Ω such that $F*\mu = F*\mathcal{H}^{N-1}\lfloor\Gamma$ in $\mathbb{R}^N\setminus\overline{\Omega}$, i.e.,

$$\int F(x-y)\,d\mu(y) = \int_\Gamma F(x-y)\,d\mathcal{H}^{N-1}(y) \quad \left(x\in\mathbb{R}^N\setminus\overline{\Omega}\right). \tag{1.2}$$

In fact, (1.2) can be replaced by the equivalent condition that

$$\int h\,d\mu = \int_\Gamma h\,d\mathcal{H}^{N-1} \tag{1.3}$$

holds for all harmonic functions h defined in a neighborhood of $\overline{\Omega}$. Indeed, it is obvious that (1.3) implies (1.2). Conversely, if Γ satisfies (1.2), then by extending each harmonic function h to be smooth and have compact support in $\mathbb{R}^N$, we see that

$$\begin{aligned}\int h(y)\,d\mu(y) &= \int_{\mathbb{R}^N} \Delta h(x)\left(\int F(y-x)\,d\mu(y)\right)dx \\ &= \int_{\mathbb{R}^N} \Delta h(x)\left(\int_\Gamma F(y-x)\,d\mathcal{H}^{N-1}(y)\right)dx \\ &= \int_\Gamma h(y)\,d\mathcal{H}^{N-1}(y).\end{aligned}$$

Thus, (1.3) follows from (1.2).

The mean value property of harmonic functions implies that (1.3) holds when $\mu = N\omega_N\delta_0$ and $\Gamma = \partial B(0,1)$, where δ_0 is the Dirac measure supported at the origin and $B(0,1)$ is the unit ball in $\mathbb{R}^N$. Thus, the identity (1.3) can be seen as a generalization of the mean value formula for harmonic functions.

From this point of view, we also consider an analogous problem: For a prescribed measure μ, find a domain Ω such that

$$\int h\,d\mu = \int_\Omega h\,dx \tag{1.4}$$

holds for all harmonic functions h defined in a neighborhood of $\overline{\Omega}$. This problem also has a physical interpretation, and it is sometimes referred to as the "Potato Kugel" problem, especially when the uniqueness of a domain Ω is concerned.

Definition 1.1. *A closed surface Γ satisfying* (1.3) *is called a quadrature surface of μ for harmonic functions. Analogously, a domain Ω satisfying* (1.4) *is called a quadrature domain of μ for harmonic functions.*

The existence of a quadrature surface Γ of a prescribed μ has been studied by several authors with different approaches. Developing the idea of super/subsolutions of Beurling [4], Henrot [12] was able to prove that the existence of Γ is guaranteed when a supersolution and a subsolution are available. Gustafsson & Shahgholian [11] followed a variational approach developed by Alt & Caffarelli [1], namely, they consider the minimization problem for the functional

$$J(u) := \int_{\mathbb{R}^N} \left(|\nabla u|^2 - 2fu + \chi_{\{u>0\}}\right) dx,$$

and proved the existence and regularity of a minimizer u. Then, u is shown to satisfy the Euler-Lagrange equation

$$-\Delta u = f\lfloor\Omega - \mathcal{H}^{N-1}\lfloor\partial\Omega, \qquad \Omega = \{u > 0\},$$

and thus $\Gamma = \partial\Omega$ is a quadrature surface of μ with $d\mu = f\,dx$.

Similarly, a quadrature domain has a variational characterization and can be obtained by solving an obstacle problem (see Sakai [18] and Gustafsson [10] for the detail). Moreover, the uniqueness of a quadrature domain follows from an argument based on the maximum principle. Indeed, it was shown by Sakai [17] that, if a quadrature domain Ω satisfies

$$F * (\mu - \chi_\Omega) > 0$$

everywhere in Ω, then there is no quadrature domain other than Ω. The above condition can be verified, in particular, when μ concentrates, relative to Ω.

However, as pointed out by Henrot [12], the uniqueness of a quadrature surface cannot be expected in general. He showed an example that the number of connected quadrature surfaces of $\mu(t) := t\delta_{(1,0)} + t\delta_{(-1,0)}$ in $\mathbb{R}^2$ changes according to the value of $t > 0$. The collapse of the uniqueness seems to indicate a bifurcation phenomenon of solutions to (1.3) with a parametrized measure $\mu = \mu(t)$. Hence, toward understanding of the uniqueness issue, we need to consider the corresponding family of surfaces $\Gamma = \Gamma(t)$. In this respect, it is natural to ask if there is a "flow" for surfaces $\{\Gamma(t)\}_{t>0}$ such that each $\Gamma(t)$ is a quadrature surface of a given parametrized measure $\mu(t)$. As a matter of fact, when $\mu(t) = t\delta_0 + \chi_{\Omega(0)}$ and $\Omega(t)$ is the corresponding quadrature domain, it is known that the Hele-Shaw flow, a model of interface dynamics in fluid mechanics, plays the desired role. This surprising connection between the two different physical problems was discovered by Richardson [16]. From this fact, the investigation of the evolution of quadrature domains is reduced to that of the Hele-Shaw flow, and the latter has been successfully proceeded by complex analysis and several methods in partial differential equations.

We are thus motivated to derive a flow having the corresponding property for quadrature surfaces, and eventually arrive at the following geometric flow:

$$\begin{aligned} &v_n = p \quad \text{for } x \in \partial\Omega(t), \\ &\text{where } \begin{cases} -\Delta p = \mu & \text{for } x \in \Omega(t), \\ (N-1)Hp + \dfrac{\partial p}{\partial n} = 0 & \text{for } x \in \partial\Omega(t), \end{cases} \end{aligned} \tag{1.5}$$

where v_n is the growing speed of $\partial\Omega(t)$ in the outer normal direction and H is the mean curvature of $\partial\Omega(t)$. Here and in what follows, μ denotes a finite positive Radon measure with compact support in $\Omega(0)$. Note that, for each fixed time $t > 0$, the maximum principle applied to the elliptic boundary problem in (1.5) yields that $p > 0$ everywhere on $\partial\Omega(t)$ if H is positive (see the proof of (2.2) in the next section). In other words, $\Omega(t)$ expands monotonically as long as the mean curvature of $\partial\Omega(t)$ is positive.

The following theorem shows that, as desired, for a given $\partial\Omega(0)$ as initial surface, the solution to (1.5) turns out to be a one-parameter family of quadrature surfaces. Moreover, we will see that (1.5) is the only possible flow having this property. Here, we call $\{\partial\Omega(t)\}_{0\le t<T}$ a $C^{3+\alpha}$ family of surfaces if each $\partial\Omega(t)$ is of $C^{3+\alpha}$ and its time derivative is of $C^{2+\alpha}$, namely, $\partial\Omega(t)$ can be locally represented as a graph of a function in the Hölder space $C^{3+\alpha}$ and its time derivative is in $C^{2+\alpha}$ (see Section 3).

Theorem 1.2. *Let $\{\partial\Omega(t)\}_{0\le t<T}$ be a $C^{3+\alpha}$ family of surfaces, and assume that each $\partial\Omega(t)$ has positive mean curvature. Then, each $\partial\Omega(t)$ is a quadrature surface of $\mu(t) := t\mu + \mathcal{H}^{N-1}\lfloor\partial\Omega(0)$, i.e.,*

$$\int_{\partial\Omega(0)} h\, d\mathcal{H}^{N-1} + t\int h\, d\mu = \int_{\partial\Omega(t)} h\, d\mathcal{H}^{N-1} \tag{1.6}$$

holds for all harmonic functions h defined in a neighborhood of $\overline{\Omega(t)}$, if and only if $\{\partial\Omega(t)\}_{0\le t<T}$ is a solution to (1.5).

Remark 1.3. The exponent $3+\alpha$ naturally arises in the context of the Schauder theory for the oblique derivative problem (see Gilbarg & Trudinger [9]). Indeed, the regularity $H \in C^{1+\alpha}$ of the coefficient function H in the boundary condition is required for the existence of a solution $p \in C^{2+\alpha}(\overline{\Omega(t)})$ to the elliptic equation in (1.5). This implies that $\partial\Omega(t)$ is of $C^{3+\alpha}$. It is worth noting that, by taking appropriate coordinates, v_n can be regarded as the time derivative of a local function representation of $\partial\Omega(t)$. Hence, it is natural to impose the same regularity as $v_n = p \in C^{2+\alpha}$ on the time derivative of $\partial\Omega(t)$.

At this point, we are led to a fundamental question: Does the equation (1.5) really possess a unique smooth solution? The following theorem affirmatively answers this question. Here, $\{\partial\Omega(t)\}_{0\le t<T}$ is called a $h^{3+\alpha}$ solution if it is a $h^{3+\alpha}$ family of surfaces and satisfies (1.5), where $h^{3+\alpha}$ is the so-called little Hölder space and is defined as the closure of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions in the topology of the Hölder space $C^{3+\alpha}$. Since our argument relies on the theory of maximal regularity of Da Prato and Grisvard [5], it is necessary to use $h^{3+\alpha}$, characterized as a continuous interpolation space, instead of $C^{3+\alpha}$.

Theorem 1.4. *There exists a unique $h^{3+\alpha}$ solution $\{\partial\Omega(t)\}_{0\le t<T}$ to* (1.5) *for any $h^{3+\alpha}$ initial surface $\partial\Omega(0)$ with positive mean curvature.*

Let us plot the points $(\Gamma, t) \in h^{3+\alpha}\times\mathbb{R}$ if Γ is a quadrature surface of $\mu(t)$. Theorem 1.4 shows that such points form a curve

$$t \longmapsto (\partial\Omega(t), t) \quad (t \in [0,T))$$

in $h^{3+\alpha}\times\mathbb{R}$ starting from $(\partial\Omega(0),0)$, if $\partial\Omega(0)$ has positive mean curvature. Moreover, as the parameter t increases, the curve does not split into two curves from any point $(\partial\Omega(t),t)$ unless $\partial\Omega(t)$ loses the positiveness of the mean curvature.

Corollary 1.5. *There is no curve*

$$s \longmapsto (\Gamma(s), t(s)) \quad (s \in [0,\varepsilon))$$

of an $h^{3+\alpha}$ family of quadrature surfaces such that $(\Gamma(0),t(0)) = (\partial\Omega(0),0)$, $\Gamma(s) \neq \partial\Omega(t(s))$ for $0<s<\varepsilon$, and $t'(0)\ge 0$.

This paper is organized as follows. In Section 2 we prove Theorem 1.2, namely, we characterize (1.5) as a flow which produces a family of quadrature surfaces. Section 3 is devoted to proving Theorem 1.4. For this purpose, we reformulate the problem into an evolution equation in an infinite-dimensional Banach space, and proceed to the spectral analysis of the linearized operator. Finally, in section 4, we prove Corollary 1.5.

2 Generation of quadrature surfaces

In this section we show that the geometric flow (1.5) generates a family of quadrature surfaces.

We begin with a simple observation that the geometric flow remains unchanged by replacing the measure μ by the mollified measure $\tilde{\mu} := \eta_\varepsilon * \mu$, where η_ε is the standard symmetric mollifier supported on $\overline{B(0,\varepsilon)}$. Note that $\tilde{\mu}$ is then a smooth function supported in $\Omega(0)$ by taking $\varepsilon > 0$ small.

Lemma 2.1. *Let $\{\partial\Omega(t)\}_{0\le t<T}$ be a $C^{3+\alpha}$ solution to (1.5), and let $\{\partial\widetilde{\Omega(t)}\}_{0\le t<T}$ be a $C^{3+\alpha}$ solution to (1.5) with μ replaced by $\tilde{\mu}$ with the same initial surface $\partial\widetilde{\Omega(0)} = \partial\Omega(0)$. Assume moreover that $\partial\Omega(t)$ and $\partial\widetilde{\Omega(t)}$ have positive mean curvature. Then, $\partial\Omega(t) = \partial\widetilde{\Omega(t)}$ for all $0 < t < T$.*

Proof. It suffices to show that the boundary value of the solution p to the elliptic boundary problem

$$\begin{cases} -\Delta p = \mu & \text{for } x \in \Omega, \\ b_1(x)p + b_2(x)\dfrac{\partial p}{\partial n} = 0 & \text{for } x \in \partial\Omega \end{cases}$$

coincides with that of the solution $\tilde{p}$ to

$$\begin{cases} -\Delta \tilde{p} = \tilde{\mu} & \text{for } x \in \Omega, \\ b_1(x)\tilde{p} + b_2(x)\dfrac{\partial \tilde{p}}{\partial n} = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $b_1(x)$, $b_2(x)$ are positive functions on $\partial\Omega$ and $\operatorname{supp}\mu \subset \operatorname{supp}\tilde{\mu} \subset \Omega$.

To this end, we prove that $q := p - \tilde{p}$ vanishes outside $\operatorname{supp}\tilde{\mu}$. Let us decompose $q = F * (\mu - \tilde{\mu}) + h$, where F is the fundamental solution of $-\Delta$ (see (1.1)) and h is a harmonic function satisfying

(2.1)
$$\begin{cases} -\Delta h = 0 & \text{for } x \in \Omega, \\ b_1(x)h + b_2(x)\dfrac{\partial h}{\partial n} = -b_1(x)F * (\mu - \tilde{\mu}) - b_2(x)\dfrac{\partial F * (\mu - \tilde{\mu})}{\partial n} & \text{for } x \in \partial\Omega. \end{cases}$$

Then, it follows from the mean value property of harmonic functions that $F * (\mu - \tilde{\mu})$ vanishes outside $\operatorname{supp}\tilde{\mu}$. Hence, the unique solvability of the oblique derivative problem (2.1) yields that $h \equiv 0$, which completes the proof. $\square$

We now proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let us first confirm that the positiveness of the mean curvature implies that

$$v_n = p > 0 \tag{2.2}$$

everywhere on $\partial\Omega(t)$ for all $0 \leq t < T$. To see this, suppose that $p(\zeta_{\min}) = \min_{\zeta\in\partial\Omega(t)} p(\zeta) \leq 0$ for some $0 \leq t < T$ and $\zeta_{\min} \in \partial\Omega(t)$, and derive a contradiction. By the maximum principle applied to the elliptic equation in (1.5), we see that $p(\zeta_{\min}) < p(x)$ for all $x \in \Omega(t)$. Hence, from the Hopf boundary point lemma it follows that

$$(N-1)Hp(\zeta_{\min}) + \frac{\partial p}{\partial n}(\zeta_{\min}) < 0,$$

which violates the boundary condition. Note that (2.2) implies $\Omega(s) \subset \Omega(t)$ for $0 \leq s \leq t$.

Now recall that, by Lemma 2.1, we may replace the measure μ by $\tilde{\mu}$ in the equation (1.5). For each harmonic function h defined in a neighborhood of $\Omega(t)$, it follows from the well-known variational formulas for moving surfaces and domains that

$$\begin{aligned}
\frac{d}{dt}\left[\int_{\partial\Omega(t)} h\, d\mathcal{H}^{N-1}\right] &= \int_{\partial\Omega(t)} \frac{\partial h}{\partial n} v_n\, d\mathcal{H}^{N-1} + (N-1)\int_{\partial\Omega(t)} hHv_n\, d\mathcal{H}^{N-1} \\
&= \int_{\partial\Omega(t)} \left\{\frac{\partial h}{\partial n}p + (N-1)hHp\right\} d\mathcal{H}^{N-1} \\
&= \int_{\Omega(t)} (\Delta hp - h\Delta p)\, dx + \int_{\partial\Omega(t)} \left\{h\frac{\partial p}{\partial n} + (N-1)hHp\right\} d\mathcal{H}^{N-1} \\
&= \int_{\Omega(t)} h\tilde{\mu}\, dx \\
&= \int h\, d\mu,
\end{aligned}$$

where the last equality follows from the mean value property of harmonic functions. The integration with respect to t yields the identity (1.6).

Let us prove the converse statement. Differentiating the identity (1.6) with respect to t, we obtain that

$$\int h\, d\mu = \int_{\partial\Omega(t)} \left\{\frac{\partial h}{\partial n} + (N-1)hH\right\} v_n\, d\mathcal{H}^{N-1}.$$

On the other hand, denoting p by a unique solution to the elliptic equation in (1.5), we have

$$\int h\, d\mu = \int_{\partial\Omega(t)} \left\{\frac{\partial h}{\partial n} + (N-1)hH\right\} p\, d\mathcal{H}^{N-1}.$$

Hence,

$$\int_{\partial\Omega(t)} \left\{ \frac{\partial h}{\partial n} + (N-1)hH \right\} (v_n - p)\, d\mathcal{H}^{N-1} = 0 \tag{2.3}$$

must hold for any harmonic function h defined in a neighborhood of $\overline{\Omega(t)}$. Let us denote by $h_0 \in C^{2+\alpha}(\overline{\Omega(t)})$ a unique solution to

$$\begin{cases} -\Delta h_0 = 0 & \text{for } x \in \Omega(t), \\ (N-1)Hh_0 + \dfrac{\partial h_0}{\partial n} = v_n - p & \text{for } x \in \partial\Omega(t). \end{cases}$$

If h_0 can be harmonically extended to a neighborhood of $\overline{\Omega(t)}$, then substituting $h = h_0$ into (2.3) deduces that $v_n = p$. But it is not the case in general, so let us take a sequence of solutions h_k to

$$\begin{cases} -\Delta h_k = 0 & \text{for } x \in \Omega_k, \\ (N-1)H_k h_k + \dfrac{\partial h_k}{\partial n} = q & \text{for } x \in \partial\Omega_k, \end{cases}$$

where $\Omega_k \supset \overline{\Omega(t)}$ is a sequence of bounded domains such that $\partial\Omega_k$ approaches $\partial\Omega(t)$ in the $C^{3+\alpha}$ sense, H_k is the mean curvature of $\partial\Omega_k$, and q is a $C^{1+\alpha}$-extension of the function $v_n - p$ on $\partial\Omega(t)$ to $\mathbb{R}^N$, i.e., $q\lfloor_{\partial\Omega(t)} = v_n - p$. Then, the elliptic estimate

$$\|h_k\|_{C^{2+\alpha}(\overline{\Omega_k})} \le C\left(\|h_k\|_{C^0(\overline{\Omega_k})} + \|q\|_{C^{1+\alpha}(\mathbb{R}^N)}\right) \le C\|q\|_{C^{1+\alpha}(\mathbb{R}^N)} \tag{2.4}$$

holds uniformly in $k = 1, 2, \ldots$, where the second inequality follows from the fact that

$$\|h_k\|_{C^0(\overline{\Omega_k})} \le \max_{\partial\Omega_k} |h_k| \le \frac{\max_{\partial\Omega_k} |q|}{(N-1)\min_{\partial\Omega_k} H_k}. \tag{2.5}$$

The proof of (2.5) is similar to that of (2.2). Now it can be shown by (2.4) together with the mean value theorem that

$$\sup_{\partial\Omega(t)} \left| \left\{ (N-1)Hh_k + \frac{\partial h_k}{\partial n} \right\} - (v_n - p) \right| \to 0.$$

Therefore, by taking $h = h_k$ with large k, we see that the identity (2.3) cannot hold unless $v_n = p$ on $\partial\Omega(t)$. $\square$

Remark 2.2. The identity (1.6) is still valid for subharmonic functions h by replacing equality with inequality $\le$. Indeed, this follows from the positivity of p in $\Omega(t)$.

3 Existence of a solution to the geometric flow

In this section we describe the outline of the proof of Theorem 1.4. The complete proof can be found in Onodera [15], where a generalized flow which includes our flow (1.5) and the Hele-Shaw flow as special cases is studied. A direct method of the mathematical treatment of a geometric equation, which we will follow, is to reformulate the problem to a fixed boundary problem by using a time-dependent diffeomorphism such that the moving boundary transforms to a fixed reference boundary. Such a transformation makes clear the nonlinear nature of the original problem. Indeed, after the transformation, we encounter the situation where the evolution equation with fixed boundary turns out to be fully-nonlinear. The theory of maximal regularity of Da Prato and Grisvard [5] enables us to handle fully-nonlinear abstract parabolic equations by taking a continuous interpolation space as phase space. Thus, our effort will be made mainly to prove the "parabolicity" of the equation, namely, that the linearized operator is an infinitesimal generator of a strongly continuous analytic semigroup.

3.1 Reduction to an evolution equation

As a first step, let us reformulate the problem to an evolution equation in an abstract setting.

We fix a bounded reference domain Ω with smooth boundary Γ, and take a subdomain Ω_{sub} such that $\operatorname{supp}\mu \subset \Omega_{\mathrm{sub}} \subset \overline{\Omega_{\mathrm{sub}}} \subset \Omega$. Let us recall that the little Hölder space $h^{k+\alpha}(\overline{\Omega})$ is defined as the closure of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ (restricted to Ω) in the topology of $C^{k+\alpha}(\overline{\Omega})$. The little Hölder space $h^{k+\alpha}(\Gamma)$ on the surface Γ can also be defined in the same manner in terms of its local coordinates. Let us define

$$\mathcal{U} = \mathcal{U}_a := \{\rho \in h^{3+\alpha}(\Gamma) \mid \|\rho\|_{C^1} < a\}$$

with $a > 0$ being sufficiently small such that $\theta(\zeta, r) := \zeta + r n_0(\xi)$ defines a diffeomorphism between $\Gamma \times (-a, a)$ and its image though θ, where $n_0(\zeta)$ is the unit outer normal vector at $\zeta \in \Gamma$. In particular, for any $\rho \in \mathcal{U}$,

$$\Gamma_\rho := \{\zeta + \rho(\zeta) n_0(\zeta) \in \mathbb{R}^N \mid \zeta \in \Gamma\} \tag{3.1}$$

defines a $h^{3+\alpha}$ surface diffeomorphic to Γ though the diffeomorphism $\theta_\rho(\zeta) := \theta(\zeta, \rho(\zeta)) = \zeta + \rho(\zeta) n_0(\zeta)$ from Γ to Γ_ρ.

For the precise descriptions of the outer unit normal vector field n_ρ on Γ_ρ and a diffeomorphism from Ω to Ω_ρ, where Ω_ρ is the domain enclosed by Γ_ρ, we will use a level set representation of the surface Γ_ρ. Let us denote by ζ_0 and r_0 the components of the inverse map θ^{-1} such that $\theta^{-1}(x) = (\zeta_0(x), r_0(x))$. Note that $\zeta_0(x)$ is the nearest point on Γ to the point x, and $r_0(x)$ is the signed distance from Γ to x. It is then easy to see that

$$L_\rho(x) := r_0(x) - \rho(\zeta_0(x)) \quad (x \in \theta(\Gamma \times (-a, a)))$$

defines Γ_ρ as its 0-level set. This representation is now used to define the normal vector field $n_\rho \in h^{3+\alpha}(\Gamma, \mathbb{R}^N)$ and a diffeomorphism from Ω to Ω_ρ, which we denote again by θ_ρ, as follows:

$$n_\rho(\zeta) := \frac{\nabla L_\rho(\theta_\rho(\zeta))}{|\nabla L_\rho(\theta_\rho(\zeta))|},$$
$$\theta_\rho(x) := \begin{cases} \theta\left(\zeta_0(x), r_0(x) + \varphi(r_0(x))\rho(\zeta_0(x))\right) & (x \in \theta(\Gamma \times (-a,a))), \\ x & (x \notin \theta(\Gamma \times (-a,a))), \end{cases}$$

where φ is a smooth cut-off function satisfying

$$\varphi(r) := \begin{cases} 1 & (|r| \le a/4), \\ 0 & (|r| \ge 3a/4) \end{cases} \quad \text{and} \quad \left|\frac{d\varphi}{dr}(r)\right| < \frac{4}{a}.$$

We also note that the speed v_n of the moving boundary at $\theta_\rho(\zeta) \in \Gamma_\rho$ can be represented by $(\partial\rho/\partial t)(\zeta)/|\nabla L_\rho(\theta_\rho(\zeta))|$.

The pull-back and push-forward operators induced by θ_ρ are defined by

$$\theta_\rho^* u := u \circ \theta_\rho, \quad \theta_*^\rho v := v \circ \theta_\rho^{-1}$$

for $u \in h^{k+\alpha}(\overline{\Omega_\rho})$, $v \in h^{k+\alpha}(\overline{\Omega})$, respectively. Then it can be shown that θ_ρ^*, θ_*^ρ are isomorphisms between $h^{k+\alpha}(\overline{\Omega_\rho})$ and $h^{k+\alpha}(\overline{\Omega})$, and $(\theta_\rho^*)^{-1} = \theta_*^\rho$. In the same fashion, θ_ρ^*, θ_*^ρ also denote isomorphisms between $h^{k+\alpha}(\Gamma_\rho)$ and $h^{k+\alpha}(\Gamma)$.

Given $\rho \in \mathcal{U}$, we now define transformed operators $A(\rho)$, $B(\rho)$ and $R(\rho)$ by

$$A(\rho) := \theta_\rho^*(-\Delta)\theta_*^\rho,$$
$$B(\rho)v := \operatorname{Tr}\theta_\rho^*\langle\nabla\theta_*^\rho v, n_\rho\rangle,$$
$$R(\rho)v := (N-1)M_{H(\rho)}\operatorname{Tr} v + B(\rho)v,$$

where Tr and M_ψ are the trace operator and the pointwise multiplication operator defined by

$$\operatorname{Tr} v(\zeta) := v(\zeta), \quad (M_\varphi\psi)(\zeta) := \varphi(\zeta)\psi(\zeta) \quad (\zeta \in \Gamma)$$

for $v \in h^{k+\alpha}(\overline{\Omega})$ and $\varphi, \psi \in h^{k+\alpha}(\Gamma)$, respectively, and $H(\rho) \in h^{1+\alpha}(\Gamma)$ assigns the mean curvature of Γ_ρ at $\theta_\rho(\zeta)$ to the point $\zeta \in \Gamma$. Note also that here we have used the notation $\langle\cdot,\cdot\rangle$ to denote the pointwise inner product. It can be shown (see Escher & Simonett [7, 8]) that

$$A \in C^\omega\left(\mathcal{U}, \mathcal{L}\left(h^{2+\alpha}(\overline{\Omega}), h^\alpha(\overline{\Omega})\right)\right),$$
$$B \in C^\omega\left(\mathcal{U}, \mathcal{L}\left(h^{2+\alpha}(\overline{\Omega}), h^{1+\alpha}(\Gamma)\right)\right),$$
$$R \in C^\omega\left(\mathcal{U}, \mathcal{L}\left(h^{2+\alpha}\left(\overline{\Omega} \setminus \Omega_{\text{sub}}\right), h^{1+\alpha}(\Gamma)\right)\right).$$

In view of (3.1), the moving surface $\partial\Omega(t)$ can be represented by $\rho(t) = \rho(\cdot, t)$ which is a real-valued function defined on the fixed reference surface Γ. Hence, the problem can be reduced to the following system of differential equations, in which unknowns are the functions ρ and u:

$$\partial_t \rho = M_{|\theta_\rho^*(\nabla L_\rho)|}\mathrm{Tr}\,\left(\theta_\rho^* E + u\right) \tag{3.2}$$

$$\text{where } \begin{cases} A(\rho)u = 0, \\ R(\rho)u = -R(\rho)\theta_\rho^* E. \end{cases} \tag{3.3}$$

Here, E is defined by

$$E(x) = E_\mu(x) := (F * \mu)(x),$$

and hence $-\Delta E = \mu$.

Furthermore, since u is determined only by ρ by virtue of the unique solvability of the elliptic equation (3.3) (see Gilbarg and Trudinger [9, Theorem 6.31]), the problem becomes a non-local evolution equation. To make it precise, let us define

$$\begin{array}{ll} S : \mathcal{U} \to \mathcal{L}(h^\alpha(\overline{\Omega}), h^{2+\alpha}(\overline{\Omega})), & S(\rho)v := (A(\rho), R(\rho))^{-1}(v, 0), \\ T : \mathcal{U} \to \mathcal{L}(h^{1+\alpha}(\Gamma), h^{2+\alpha}(\overline{\Omega})), & T(\rho)\varphi := (A(\rho), R(\rho))^{-1}(0, \varphi). \end{array}$$

Then, we see that $u = -T(\rho)R(\rho)\theta_\rho^* E$. Therefore, our problem is to solve the following evolution equation:

$$\partial_t \rho + \Phi(\rho) = 0, \tag{3.4}$$

where

$$\Phi : \mathcal{U} \to h^{1+\alpha}(\Gamma), \quad \Phi(\rho) := M_{|\theta_\rho^*(\nabla L_\rho)|}\mathrm{Tr}\,\left(T(\rho)R(\rho) - I\right)\theta_\rho^* E.$$

Here, I is the identity map.

3.2 Linearized operator and its principal part

The theory of abstract evolution equations enables us to reduce the existence of a solution of (3.4) to the spectral properties of the linearized operator $\partial\Phi(\rho)$ of Φ at $\rho \in \mathcal{U}$. Indeed, once $\partial\Phi(\rho)$ is shown to be a sectorial operator, i.e., an infinitesimal generator of an analytic semigroup, then it follows from the theory of maximal regularity of Da Prato and Grisvard [5] that the equation (3.4) is uniquely solvable for initial data in a certain function space characterized as a continuous interpolation space.

By the implicit function theorem, we have the representation of the linearized operator $\partial T(\rho)$ of T at $\rho \in \mathcal{U}$ as follows.

Lemma 3.1. *For $\rho \in \mathcal{U}$ and $\varphi \in h^{1+\alpha}(\Gamma)$, let us set $v = v(\rho) := T(\rho)\varphi$, i.e., v satisfies*

$$\begin{cases} A(\rho)v = 0 & \text{in } \Omega, \\ R(\rho)v = \varphi & \text{on } \partial\Omega. \end{cases}$$

Then, the linearized operator $\partial v(\rho) \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\overline{\Omega}))$ *of* v *at* ρ *is given by*

$$\partial v(\rho)[\tilde{\rho}] = \partial\left(T(\rho)\varphi\right)[\tilde{\rho}] = -S(\rho)\partial A(\rho)[\tilde{\rho}]T(\rho)\varphi - T(\rho)\partial R(\rho)[\tilde{\rho}]T(\rho)\varphi.$$

Moreover, $T \in C^\omega(\mathcal{U}, \mathcal{L}(h^{1+\alpha}(\Gamma), h^{2+\alpha}(\overline{\Omega})))$.

From the above lemma, we see that

$$\partial\Phi(\rho)[\tilde{\rho}] = M_{|\theta^*_\rho(\nabla L_\rho)|}\mathrm{Tr}\, T(\rho)\partial R(\rho)[\tilde{\rho}]\left(I - T(\rho)R(\rho)\right)\theta^*_\rho E + F_1(\rho)[\tilde{\rho}] + F_2(\rho)[\tilde{\rho}] + F_3(\rho)[\tilde{\rho}],$$

where the linear operators

$$\begin{aligned}
F_1(\rho)[\tilde{\rho}] &:= -M_{|\theta^*_\rho(\nabla L_\rho)|}\mathrm{Tr}\, S(\rho)\partial A(\rho)[\tilde{\rho}]T(\rho)R(\rho)\theta^*_\rho E,\\
F_2(\rho)[\tilde{\rho}] &:= \partial M_{|\theta^*_\rho(\nabla L_\rho)|}[\tilde{\rho}]\mathrm{Tr}\,\left(T(\rho)R(\rho) - I\right)\theta^*_\rho E,\\
F_3(\rho)[\tilde{\rho}] &:= M_{|\theta^*_\rho(\nabla L_\rho)|}\mathrm{Tr}\,\left(T(\rho)R(\rho) - I\right)\partial(\theta^*_\rho E)[\tilde{\rho}]
\end{aligned}$$

can be thought of as perturbations in the sense that

$$\|F_j(\rho)[\tilde{\rho}]\|_{h^{2+\alpha}(\Gamma)} \le C\|\tilde{\rho}\|_{h^{2+\alpha}(\Gamma)} \quad (j = 1, 2, 3),$$

where the constant C depends on $\rho \in \mathcal{U}$, but not on $\tilde{\rho} \in h^{3+\alpha}(\Gamma)$.

Moreover, the operator $\partial R(\rho)$ can also be decomposed further into the principal part and its perturbation. For this purpose, let us recall that the mean curvature operator $H = H(\rho)$ has a useful representation as in the following lemma. Here we take γ such that $\alpha < \gamma < 1$ and set

$$\mathcal{V} = \mathcal{V}_a := \{\rho \in h^{2+\gamma}(\Gamma) \mid \|\rho\|_{C^1} < a\}.$$

Lemma 3.2 (Escher & Simonett [7, Lemma 3.1]). *For each* $\rho \in \mathcal{U}$*, the mean curvature operator* $H(\rho)$ *can be decomposed as*

$$H(\rho) = P(\rho)\rho + K(\rho),$$

where $P \in C^\omega\left(\mathcal{V}, \mathcal{L}\left(h^{3+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)\right)\right)$ *and* $K \in C^\omega(\mathcal{V}, h^{1+\gamma}(\Gamma))$.

Hence, for $v \in h^{2+\alpha}\left(\overline{\Omega} \setminus \Omega_{\mathrm{sub}}\right)$, we have

$$\partial\left(R(\rho)v\right)[\tilde{\rho}] = (N-1)M_v P(\rho)[\tilde{\rho}] + F_4(\rho, v)[\tilde{\rho}],$$

where

$$\|F_4(\rho, v)[\tilde{\rho}]\|_{h^{1+\alpha}(\Gamma)} \le C\|v\|_{h^{2+\alpha}(\Gamma)}\|\tilde{\rho}\|_{h^{2+\gamma}(\Gamma)}$$

with C being a constant independent of $\tilde{\rho}$. Therefore, the linearized operator $\partial\Phi(\rho)$ can now be represented in the following form:

$$\partial\Phi(\rho)[\tilde{\rho}] = (N-1)M_1(\rho)\mathrm{Tr}\, T(\rho)M_2(\rho)P(\rho)[\tilde{\rho}] + F(\rho)[\tilde{\rho}],$$

where

$$\begin{aligned}
M_1(\rho) &:= M_{|\theta^*_\rho(\nabla L_\rho)|} \in \mathcal{L}\left(h^{2+\alpha}(\Gamma)\right),\\
M_2(\rho) &:= M_{(I - T(\rho)R(\rho))\theta^*_\rho E} \in \mathcal{L}\left(h^{1+\alpha}(\Gamma)\right),\\
F(\rho) &\in \mathcal{L}\left(h^{2+\gamma}(\Gamma), h^{2+\alpha}(\Gamma)\right).
\end{aligned}$$

3.3 The generation property of the linearized operator

Our task is now to prove that the linear operator

$$W = W(\rho) := -M_1(\rho)\mathrm{Tr}\, T(\rho)M_2(\rho)P(\rho) \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\Gamma))$$

is sectorial in $h^{2+\alpha}(\Gamma)$, i.e., it generates an analytic semigroup on $h^{2+\alpha}(\Gamma)$. Indeed, a standard perturbation result of sectorial operators implies that, if W is sectorial, then $-\partial\Phi(\rho)$ is also sectorial. The following theorem is the main assertion in this section.

Theorem 3.3. *$W \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\Gamma))$ is sectorial in $h^{3+\alpha}(\Gamma)$.*

Corollary 3.4. *$-\partial\Phi(\rho) \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\Gamma))$ is sectorial in $h^{3+\alpha}(\Gamma)$.*

To prove Theorem 3.3, it is well-known (see Amann [2]) that W is sectorial if there exist positive constants λ_* and C such that

(i) $\lambda_* I - W \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\Gamma))$ is bijective, i.e., λ_* is in the resolvent set.

(ii) $|\lambda|\|\tilde{\rho}\|_{h^{2+\alpha}(\Gamma)} + \|\tilde{\rho}\|_{h^{3+\alpha}(\Gamma)} \leq C\|(\lambda I - W)\tilde{\rho}\|_{h^{2+\alpha}(\Gamma)}$ holds for $\tilde{\rho} \in h^{3+\alpha}(\Gamma)$ and $\lambda \in \{z \in \mathbb{C} \mid \mathrm{Re}\, z \geq \lambda_*\}$.

Let us first confirm the condition (i) by assuming (ii). Since (ii) implies that $\lambda_* I - W$ is injective, we only need to prove that it is also surjective. Note that $\mathcal{U}$ is star-shaped with respect to 0 in $h^{3+\alpha}(\Gamma)$ and $\mathcal{K} := \{t\rho \in \mathcal{U} \mid 0 \leq t \leq 1\}$ is a compact subset in $\mathcal{U}$. Hence, from the continuity of the map $\rho \mapsto W = W(\rho)$ it follows that the constant C in the resolvent estimate (ii) can be chosen uniformly in $\rho \in \mathcal{K}$. Therefore, by the continuity method (see Gilbarg & Trudinger [9, Theorem 5.2]) together with the uniform resolvent estimate (ii), it is sufficient to show that $\lambda_* I - W$ is surjective in the case $\rho = 0$.

Then, it is known that

$$P(0) = -\frac{1}{N-1}\Delta_\pi^\Gamma, \tag{3.5}$$

where Δ_π^Γ is the principal part of the Laplace-Beltrami operator with respect to Γ. Moreover, we have

$$v := (I - T(0)R(0))\, E > 0 \tag{3.6}$$

everywhere on Γ. This can be verified in the same way as (2.2), since v satisfies

$$\begin{cases} -\Delta v = \mu, \\ R(0)v = 0. \end{cases}$$

Now (3.5) and (3.6) imply that

$$I + M_2(0)P(0) = I + M_{(I-T(0)R(0))E}P(0) \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{1+\alpha}(\Gamma))$$

is a bijective operator having bounded inverse.

Note also that

$$M_1(0)\mathrm{Tr}\, T(0) = M_{|\nabla L_0|}\mathrm{Tr}\, T(0) \in \mathcal{L}\left(h^{1+\alpha}(\Gamma), h^{2+\alpha}(\Gamma)\right)$$

is bijective. This follows from $|\nabla L_0| > 0$ and the unique solvability of the oblique derivative problem in the Hölder spaces (see Gilbarg & Trudinger [9, Theorem 6.31]).

In the expression

$$\lambda_* I - W = M_1(0)\mathrm{Tr}\, T(0)\left\{I + M_2(0)P(0)\right\} + \lambda_* I - M_1(0)\mathrm{Tr}\, T(0),$$

the second and third operators in the right hand side are compact perturbations, since the embedding $h^{3+\alpha}(\Gamma) \hookrightarrow h^{2+\alpha}(\Gamma)$ is compact. Furthermore, as we have already seen, the first one is a bijective operator from $h^{3+\alpha}(\Gamma)$ to $h^{2+\alpha}(\Gamma)$. Therefore, $\lambda_* I - W$ is a Fredholm operator of index 0. Now the assertion follows from the fact that $\lambda_* I - W$ is injective.

We will establish the remaining resolvent estimate (ii) in the following sections.

3.4 Fourier multiplier operators associated with localized operators

Let us take an atlas $\{U_l, \psi_l\}_{1\le l\le m}$ of $R_d := \theta(\Gamma \times (-d, 0])$ for small $0 < d < a/4$ such that $\mathrm{diam}\, U_l < d$ and that ψ_l maps $Q := (-d,d)^{N-1} \times [0,d)$, $Q_0 := (-d,d)^{N-1} \times \{0\}$ onto U_l, $U_l \cap \Gamma$, respectively. Note that the number of local coordinates m depends on d.

Localizing the operator W to each U_l, and choosing an appropriate constant coefficient operator on $\mathbb{R}^{N-1}$ which approximates W in that localized region U_l, we will show that this constant coefficient operator has a representation as a Fourier multiplier operator, and moreover that it generates an analytic semigroup in an appropriate Banach space, namely, the little Hölder space $h^{2+\alpha}(\mathbb{R}^{N-1})$. The latter will be established by applying a general result due to H. Amann, which states that, for given $\sigma \in \mathcal{E}ll\mathcal{S}_1^\infty(\gamma_*)$, $\gamma_* > 0$ and $\eta_0 > 0$, it follows that

$$\Sigma_{\eta_0} := -\mathcal{F}^{-1}\mathcal{M}_{\sigma(\cdot,\eta_0)}\mathcal{F} \in \mathcal{L}\left(h^{3+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1})\right)$$

is sectorial, i.e., it generates a strongly continuous analytic semigroup on $h^{2+\alpha}(\mathbb{R}^{N-1})$. Here, $\sigma \in \mathcal{E}ll\mathcal{S}_1^\infty(\gamma_*)$ if $\sigma = \sigma(\xi,\eta) \in C^\infty\left(\mathbb{R}^{N-1} \times (0,\infty)\right)$ is positively homogeneous of degree one and its all derivatives are bounded on the set $\left\{|\xi|^2 + \eta^2 = 1\right\}$ and if

$$\mathrm{Re}\,\sigma(\xi,\eta) \ge \gamma_* \sqrt{|\xi|^2 + \eta^2} \quad \left((\xi,\eta) \in \mathbb{R}^{N-1} \times (0,\infty)\right) \tag{3.7}$$

holds. The linear operator $\mathcal{M}_\phi$ with a given function ϕ on $\mathbb{R}^{N-1}$ is the localized version of the pointwise multiplication operator induced by ϕ.

Let us fix $\rho \in \mathcal{U}$ and $(U,\psi) = (U_l, \psi_l)$ for some $l = 1, \ldots, m$, and define the pull-back and push-forward operators induced by ψ by

$$\psi^* u := u \circ \psi, \quad \psi_* v := v \circ \psi^{-1}$$

for $u \in h^{k+\alpha}(\overline{U})$, $v \in h^{k+\alpha}(\overline{Q})$, respectively. We then introduce local representations $\mathcal{A}$, $\mathcal{R}$ and $\mathcal{P}$ of the operators $A(\rho)$, $R(\rho)$ and $P(\rho)$ defined by

$$\mathcal{A} := \psi^* A(\rho)\psi_*, \quad \mathcal{R} := \psi^* R(\rho)\psi_*, \quad \mathcal{P} := \psi^* P(\rho)\psi_*.$$

In what follows, for simplicity, we write

$$\partial_j := \frac{\partial}{\partial \omega_j} \quad (j = 1, \ldots, N-1), \quad \partial_N := \frac{\partial}{\partial r}.$$

As shown in Escher & Simonett [7, Lemma 3.2] and [8, Lemma 3.1], we have

$$\begin{aligned}
\mathcal{A} &= -\sum_{j,k=1}^{N} a_{jk}(\rho)\partial_j\partial_k + \sum_{j=1}^{N} a_j(\rho)\partial_j,\\
\mathcal{R} &= b_0(\rho)\mathrm{Tr} - \sum_{j=1}^{N} b_j(\rho)\mathrm{Tr}\,\partial_j,\\
\mathcal{P} &= -\sum_{j,k=1}^{N-1} p_{jk}(\rho)\partial_j\partial_k
\end{aligned}$$

where $a_{jk} \in C^\omega(\mathcal{U}, h^{2+\alpha}(Q))$, $a_j \in C^\omega(\mathcal{U}, h^{1+\alpha}(Q))$, $b_j \in C^\omega(\mathcal{U}, h^{2+\alpha}(Q_0))$ and $p_{jk} \in C^\omega(\mathcal{U}, h^{2+\alpha}(Q_0))$, and we used the same notation Tr to denote the trace operator on Q_0. Moreover, the matrices $(a_{jk}(\rho)(\omega, r))$, $(p_{jk}(\rho)(\omega))$ are symmetric and uniformly positive definite on Q, Q_0, respectively, and $b_0(\rho)$, $b_N(\rho)$ are uniformly positive on Q_0. Here, we may further assume that

$$b_j(\rho) = 0 \quad (j = 1, \ldots, N-1).$$

Indeed, the validity of this assumption is guaranteed by taking the diffeomorphisms ψ_l so that each $\theta_\rho \circ \psi_l$ preserves the normal directions to the corresponding boundaries, namely,

$$\partial_N(\theta_\rho \circ \psi_l) = D(\theta_\rho \circ \psi_l)e_N = -s\,(n_\rho \circ \psi_l)$$

holds with some positive number s at each point on Q_0, where $e_N := {}^t(0, \ldots, 0, 1)$. For the construction of such a diffeomorphism, we refer to Ni & Takagi [14].

We are now in a position to introduce associated constant coefficient operators. By setting

$$a^0_{jk} := a_{jk}(\rho)(0,0), \quad b^0_j := b_j(\rho)(0), \quad p^0_{jk} = p_{jk}(\rho)(0),$$

let us define

$$\begin{aligned}
\mathcal{A}_0 &:= -\sum_{j,k=1}^{N} a^0_{jk}\partial_j\partial_k,\\
\mathcal{R}_0 &:= b^0_0\mathrm{Tr} - b^0_N\mathrm{Tr}\,\partial_N,\\
\mathcal{P}_0 &:= I - \sum_{j,k=1}^{N-1} p^0_{jk}\partial_j\partial_k.
\end{aligned}$$

The constant coefficient operator $\mathcal{T}_0$ associated with $T(\rho)$ will be defined such that, for $\varphi \in h^{1+\alpha}(\mathbb{R}^{N-1})$, $v := \mathcal{T}_0\varphi \in h^{2+\alpha}(\mathbb{R}^{N-1} \times [0,\infty))$ and v satisfies

$$(3.8) \qquad \begin{cases} (I + \mathcal{A}_0)v = 0 & \text{in } \mathbb{R}^{N-1} \times (0,\infty), \\ \mathcal{R}_0 v = \varphi & \text{on } \mathbb{R}^{N-1} \simeq \mathbb{R}^{N-1} \times \{0\}. \end{cases}$$

To derive an explicit representation of $\mathcal{T}_0$, we set

$$z(\xi) := \frac{i}{a^0_{NN}} \sum_{j=1}^{N-1} a^0_{jN}\xi_j + \frac{1}{a^0_{NN}} \sqrt{a^0_{NN}\left(1 + \sum_{j,k=1}^{N-1} a^0_{jk}\xi_j\xi_k\right) - \left(\sum_{j=1}^{N-1} a^0_{jN}\xi_j\right)^2},$$

where $i := \sqrt{-1}$. Then, $z = z(\xi)$ is a solution to the quadratic equation

$$1 + \sum_{jk=1}^{N-1} a^0_{jk}\xi_j\xi_k + 2i\left(\sum_{j=1}^{N-1} a^0_{jN}\xi_j\right) z - a^0_{NN}z^2 = 0$$

and satisfies $\operatorname{Re} z(\xi) > 0$ by the ellipticity of (a^0_{jk}). Denoting by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the (partial) Fourier transform and the inverse (partial) Fourier transform on $\mathbb{R}^{N-1}$, respectively, we have an explicit representation formula of the solution operator $\mathcal{T}_0$ as the following lemma shows.

Lemma 3.5. *Let $\mathcal{T}_0$ be defined by*

$$\mathcal{T}_0\varphi(\omega, r) := \left[\mathcal{F}^{-1}\mathcal{M}_{\sigma_1(\cdot,r)}\mathcal{F}\varphi\right](\omega),$$
$$\sigma_1(\xi, r) := \frac{e^{-z(\xi)r}}{b^0_0 + b^0_N z(\xi)}.$$

Then, $\mathcal{T}_0 \in \mathcal{L}(h^{1+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1} \times [0,\infty)))$ and, for any $\varphi \in h^{1+\alpha}(\mathbb{R}^{N-1})$, $v := \mathcal{T}_0\varphi$ is the unique solution to (3.8) *in $h^{2+\alpha}(\mathbb{R}^{N-1} \times [0,\infty))$.*

Proof. By a direct computation, it is easy to see that $v := \mathcal{T}_0\varphi$ satisfies (3.8) for smooth φ. Moreover, $\mathcal{T}_0 \in \mathcal{L}(h^{1+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1} \times [0,\infty)))$ follows from the decomposition

$$\mathcal{T}_0\varphi(\omega, r) = \left[\left(\mathcal{F}^{-1}\mathcal{M}_{\sigma_{1,1}(\cdot,r)}\mathcal{F}\right)\left(\mathcal{F}^{-1}\mathcal{M}_{\sigma_{1,2}}\mathcal{F}\right)\right](\omega),$$

where

$$\sigma_{1,1}(\xi, r) := e^{-z(\xi)r}, \quad \sigma_{1,2}(\xi) := \left(b^0_0 + b^0_N z(\xi)\right)^{-1}.$$

Indeed, $\mathcal{F}^{-1}\mathcal{M}_{\sigma_{1,1}(\cdot,r)}\mathcal{F} \in \mathcal{L}(h^{2+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1} \times [0,\infty)))$ can be checked as in Escher & Simonett [6, Lemma B.2], and also it is easy to prove that $\mathcal{F}^{-1}\mathcal{M}_{\sigma_{1,2}}\mathcal{F} \in \mathcal{L}(h^{1+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1}))$ in view of Escher & Simonett [6, Theorem A.1]. For the uniqueness of a solution, it suffices to show that any solution $v \in h^{2+\alpha}(\mathbb{R}^{N-1} \times [0,\infty))$ of

$$\begin{cases} (I + \mathcal{A}_0)v = 0 & \text{in } \mathbb{R}^{N-1} \times (0,\infty), \\ \mathcal{R}_0 v = 0 & \text{on } \mathbb{R}^{N-1}. \end{cases}$$

must be identical with the trivial solution $v \equiv 0$. By virtue of the Phragmén-Lindelöf principle, this can be reduced to showing that $v = 0$ on the boundary $\mathbb{R}^{N-1}$. Let us prove that $v \le 0$ on $\mathbb{R}^{N-1}$ by assuming

$$c := \sup_{\omega \in \mathbb{R}^{N-1}} v(\omega, 0) > 0$$

and deriving a contradiction. For any $\omega \in \mathbb{R}^{N-1}$ and $r > 0$, observe that

$$\begin{aligned} v(\omega,0) + \frac{b_0^0}{b_N^0} r v(\omega,0) - v(\omega,r) &= v(\omega,0) + r\partial_N v(\omega,0) - v(\omega,r) \\ &= \int_0^r (\partial_N v(\omega,0) - \partial_N v(\omega,s))\, ds \\ &\le \frac{r^2}{2} \|v\|_{h^{2+\alpha}(\mathbb{R}^{N-1}\times[0,\infty))}. \end{aligned}$$

Thus, by choosing a sufficiently small $\varepsilon > 0$ and $\omega \in \mathbb{R}^{N-1}$ such that $v(\omega,0) > c - \varepsilon$, we see that

$$\begin{aligned} v(\omega,r) &\ge v(\omega,0) + \frac{b_0^0}{b_N^0} r v(\omega,0) - \frac{r^2}{2}\|v\|_{h^{2+\alpha}(\mathbb{R}^{N-1}\times[0,\infty))} \\ &> c - \varepsilon + \frac{b_0^0}{b_N^0} r(c-\varepsilon) - \frac{r^2}{2}\|v\|_{h^{2+\alpha}(\mathbb{R}^{N-1}\times[0,\infty))} \\ &> c, \end{aligned}$$

where the last inequality is valid for $\varepsilon > 0$ and $r \in (0,1)$ such that

$$r\left(\frac{b_0^0}{b_N^0} c - \frac{r}{2}\|v\|_{h^{2+\alpha}(\mathbb{R}^{N-1}\times[0,\infty))}\right) > \varepsilon\left(1 + \frac{b_0^0}{b_N^0} r\right).$$

and the existence of such a pair of ε and r can be easily checked. However, recalling that the Phragmén-Lindelöf principle yields $v(\omega,r) < c$ for all $\omega \in \mathbb{R}^{N-1}$ and $r > 0$, we are now arriving at a contradiction and thus $v \le 0$ is proved. The inequality $v \ge 0$ can be proved by a similar argument. □

For later use, we also provide the solution operator $\mathcal{S}_0$ of the following boundary value problem:

$$\begin{cases} (I + \mathcal{A}_0)v = f & \text{in } \mathbb{R}^{N-1} \times (0,\infty), \\ \mathcal{R}_0 v = 0 & \text{on } \mathbb{R}^{N-1}. \end{cases} \tag{3.9}$$

In what follows, we write $\mathcal{F}_N$ and $\mathcal{F}_N^{-1}$ for the Fourier transform and the inverse Fourier transform on $\mathbb{R}^N$, respectively, and $\mathcal{E} \in \mathcal{L}(h^\alpha(\mathbb{R}^{N-1} \times [0,\infty)), h^\alpha(\mathbb{R}^N))$ denotes an extension operator, i.e., $\mathcal{E} f = f$ on $\mathbb{R}^{N-1} \times [0,\infty)$.

Lemma 3.6. *Let $\mathcal{S}_0$ be defined by*

$$\mathcal{S}_0 f(\omega, r) := (I - \mathcal{T}_0 \mathcal{R}_0) \left\{ \mathcal{F}_N^{-1} \mathcal{M}_{\sigma_2} \mathcal{F}_N \mathcal{E} f \right\} \lfloor_{\mathbb{R}^{N-1} \times [0,1]},$$
$$\sigma_2(\xi) := \left(1 + \sum_{j,k=1}^{N} a_{jk}^0 \xi_j \xi_k \right)^{-1}.$$

Then, $\mathcal{S}_0 \in \mathcal{L}(h^\alpha(\mathbb{R}^{N-1} \times [0,\infty)), h^{2+\alpha}(\mathbb{R}^{N-1} \times [0,\infty)))$ and, for any $f \in h^\alpha(\mathbb{R}^{N-1} \times [0,\infty))$, $v := \mathcal{S}_0 f$ is the unique solution to (3.9) in $h^{2+\alpha}(\mathbb{R}^{N-1} \times [0,\infty))$.

Proof. A direct computation shows that $v := \mathcal{S}_0 f$ satisfies (3.9) for smooth f. Moreover, Lemma 3.5 and the facts that

$$\mathcal{R}_0 \in \mathcal{L}(h^{2+\alpha}(\mathbb{R}^{N-1} \times [0,\infty)), h^{1+\alpha}(\mathbb{R}^{N-1})),$$
$$\mathcal{F}_N^{-1} \mathcal{M}_{\sigma_2} \mathcal{F}_N \in \mathcal{L}(h^\alpha(\mathbb{R}^N), h^{2+\alpha}(\mathbb{R}^N))$$

yield the desired conclusion $\mathcal{S}_0 \in \mathcal{L}(h^\alpha(\mathbb{R}^{N-1} \times [0,\infty)), h^{2+\alpha}(\mathbb{R}^{N-1} \times [0,\infty)))$. The uniqueness of a solution again follows from the Phragmén-Lindelöf principle. □

Finally, by setting

$$m_1 := \psi^* |\theta_\rho^*(\nabla L_\rho)|(0,0) > 0,$$
$$m_2 := \psi^* \left\{ (I - T(\rho) R(\rho))\, \theta_\rho^* E \right\} (0,0) > 0,$$

we define $\mathcal{W}_0$ by

$$\begin{aligned} \mathcal{W}_0 :=& - m_1 m_2 \operatorname{Tr} \mathcal{T}_0 \mathcal{P}_0 \\ =& - \mathcal{F}^{-1} \mathcal{M}_\sigma \mathcal{F}, \end{aligned}$$

where

$$\sigma(\xi) := \frac{m_1 m_2 \left(1 + \sum_{j,k=1}^{N-1} p_{jk}^0 \xi_j \xi_k \right)}{b_0^0 + b_N^0 z(\xi)}.$$

Then, we have the following proposition.

Proposition 3.7. $\mathcal{W}_0 \in \mathcal{L}\left(h^{3+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1})\right)$ *is sectorial.*

Proof. Let us define the parametrized symbol $\tilde{\sigma}$ by

$$\tilde{\sigma}(\xi, \eta) := \frac{m_1 m_2 \left(\eta^2 + \sum_{j,k=1}^{N-1} p_{jk}^0 \xi_j \xi_k \right)}{b_0^0 \eta + b_N^0 \tilde{z}(\xi, \eta)},$$

where

$$\tilde{z}(\xi, \eta) := \frac{i}{a_{NN}^0} \sum_{j=1}^{N-1} a_{jN}^0 \xi_j + \frac{1}{a_{NN}^0} \sqrt{a_{NN}^0 \left(\eta^2 + \sum_{j,k=1}^{N-1} a_{jk}^0 \xi_j \xi_k \right) - \left(\sum_{j=1}^{N-1} a_{jN}^0 \xi_j \right)^2}.$$

Note that $\tilde{z}(\xi, 1) = z(\xi)$ and hence $\tilde{\sigma}(\xi, 1) = \sigma(\xi)$. We show that $\tilde{\sigma} \in \mathcal{E}ll\mathcal{S}_1^\infty(\gamma_*)$ with some positive number γ_*. Indeed, it is easy to see that $\tilde{\sigma} \in C^\infty(\mathbb{R}^{N-1} \times (0, \infty))$, and it is positively homogeneous of degree one, and its all derivatives are bounded on $\{|\xi|^2 + \eta^2 = 1\}$. To check the condition (3.7), let a_*, p_* denote the ellipticity constants for $\mathcal{A}_0$, $\mathcal{P}_0$, i.e.,

$$\sum_{j,k=1}^{N-1} a_{jk}^0 \xi_j \xi_k + 2\tilde{\eta} \sum_{j=1}^{N-1} a_{jN}^0 \xi_j + a_{NN}^0 \tilde{\eta}^2 \geq a_* \left(|\xi|^2 + \tilde{\eta}^2 \right), \tag{3.10}$$

$$\sum_{j,k=1}^{N-1} p_{jk}^0 \xi_j \xi_k \geq p_* |\xi|^2. \tag{3.11}$$

Then, in particular, by taking $\tilde{\eta} = -(a_{NN}^0)^{-1} \sum_{j=1}^{N-1} a_{jN}^0 \xi_j$ in (3.10), we have

$$\sum_{j,k=1}^{N-1} a_{jk}^0 \xi_j \xi_k - \frac{1}{a_{NN}^0} \left(\sum_{j=1}^{N-1} a_{jN}^0 \xi_j \right)^2 \geq a_* |\xi|^2,$$

and hence

$$\begin{aligned} \operatorname{Re} \tilde{z}(\xi, \eta) &\geq \frac{1}{a_{NN}^0} \sqrt{a_{NN}^0 \left(\eta^2 + a_* |\xi|^2 \right)} \\ &\geq \sqrt{\frac{\min\{1, a_*\}}{a_{NN}^0}} \sqrt{|\xi|^2 + \eta^2} \end{aligned} \tag{3.12}$$

We also observe that

$$\begin{aligned} \left| b_0^0 \eta + b_N^0 \tilde{z}(\xi, \eta) \right|^2 &\leq 2 b_0^0 \eta^2 + 2 b_N^0 \left| \tilde{z}(\xi, \eta) \right|^2 \\ &\leq 2 b_N^0 \left(\sum_{j,k=1}^{N-1} {a_{jk}^0}^2 \right) |\xi|^2 + 2 (b_0^0 + b_N^0) \eta^2. \end{aligned} \tag{3.13}$$

Therefore, combining (3.11), (3.12) and (3.13), we deduce that

$$\begin{aligned} \operatorname{Re} \tilde{\sigma}(\xi, \eta) &= \frac{m_1 m_2 \left(\eta^2 + \sum_{j,k=1}^{N-1} p_{j,k}^0 \xi_j \xi_k \right) \left(b_0^0 \eta + b_N^0 \operatorname{Re} \tilde{z}(\xi, \eta) \right)}{\left| b_0^0 \eta + b_N^0 \tilde{z}(\xi, \eta) \right|^2} \\ &\geq \frac{m_1 m_2 \left(\eta^2 + p_* |\xi|^2 \right) \left(b_0^0 \eta + b_N^0 \sqrt{\frac{\min\{1, a_*\}}{a_{NN}^0}} \sqrt{|\xi|^2 + \eta^2} \right)}{2 b_N^0 \left(\sum_{j,k=1}^{N-1} {a_{jk}^0}^2 \right) |\xi|^2 + 2 \left(b_0^0 + b_N^0 \right) \eta^2} \\ &\geq \gamma_* \sqrt{|\xi|^2 + \eta^2}, \end{aligned}$$

where

$$\gamma_* := \frac{m_1 m_2 b_N^0 \min\{1, p_*\} \sqrt{\min\{1, a_*\}}}{2 \sqrt{a_{NN}^0} \max\left\{ b_N^0 \left(\sum_{j,k=1}^{N-1} {a_{jk}^0}^2 \right), b_0^0 + b_N^0 \right\}} > 0.$$

Therefore, $\tilde{\sigma} \in \mathcal{E}ll\mathcal{S}_1^\infty(\gamma_*)$, and hence

$$\mathcal{W}_0 = -\mathcal{F}^{-1}\mathcal{M}_{\tilde{\sigma}(\cdot,1)}\mathcal{F}$$

is a sectorial operator on $h^{2+\alpha}(\mathbb{R}^{N-1})$. □

3.5 Resolvent estimate by a perturbation argument

Proposition 3.7 implies that the operator $\mathcal{W}_0^{(l)} = \mathcal{W}_0$, which approximates W in the localized region U_l, satisfies the resolvent estimate

$$|\lambda|\|\tilde{\rho}\|_{h^{2+\alpha}(\mathbb{R}^{N-1})} + \|\tilde{\rho}\|_{h^{3+\alpha}(\mathbb{R}^{N-1})} \leq C\|(\lambda I - \mathcal{W}_0^{(l)})\tilde{\rho}\|_{h^{2+\alpha}(\mathbb{R}^{N-1})} \tag{3.14}$$

for any $\tilde{\rho} \in h^{3+\alpha}(\mathbb{R}^{N-1})$ and $\lambda \in \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \lambda_0\}$, by taking $\lambda_0 > 0$ and $C > 0$ appropriately.

We will show that $\mathcal{W}_0^{(l)}$ indeed approximates W by taking $d > 0$ so small that the atlas $\{U_l, \psi_l\}_{1\leq l\leq m}$ of R_d becomes fine enough (see the beginning of Section 3.4) in the sense that the desired resolvent estimate

$$|\lambda|\|\tilde{\rho}\|_{h^{2+\alpha}(\Gamma)} + \|\tilde{\rho}\|_{h^{3+\alpha}(\Gamma)} \leq C\|(\lambda I - W)\tilde{\rho}\|_{h^{2+\alpha}(\Gamma)} \tag{3.15}$$

holds after patching all the local estimates together. This estimate completes the proof of Theorem 3.3.

For this purpose, we take a partition of unity $\{\phi_l\}_{l=1}^m$ associated with $\{U_l\}_{l=1}^m$ such that $\operatorname{supp}\phi_l \subset U_l$ and $\bigcup_{l=1}^m \phi_l = 1$ on $R_{d/2}$. Combining the atlas and the partition of unity, we call such a pair a localization sequence of R_d. Note that, we can choose a family of smooth cut-off functions $\{\chi_l\}_{l=1}^m$ as well as a localization sequence of R_d such that $\operatorname{supp}\chi_l \subset U_l$, $\chi_l = 1$ on $\operatorname{supp}\phi_l$ and

$$\|\chi_l\|_{0,U_l} + d^\alpha[\chi_l]_{\alpha,U_l} \leq C \tag{3.16}$$

with a positive constant C which is independent of d. Here and in what follows, we use the notation

$$\|v\|_{k+\alpha,U} := \|v\|_{h^{k+\alpha}(U)}, \quad [v]_{\alpha,U} := \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|v(x)-v(y)|}{|x-y|^\alpha},$$
$$\|v\|_{k+\alpha} := \|v\|_{k+\alpha,\mathbb{R}^{N-1}}, \quad [v]_\alpha := [v]_{\alpha,\mathbb{R}^{N-1}}.$$

Now we state the following perturbation result.

Lemma 3.8. *For any $\varepsilon > 0$, $0 < \beta < \alpha$ and $\rho \in \mathcal{U}$, there are $d > 0$, a localization sequence of R_d, and a constant $C = C(\varepsilon, \beta, \rho, d)$ such that*

$$\left\|\psi_l^*(\phi_l W\tilde{\rho}) - \mathcal{W}_0^{(l)}\psi_l^*(\phi_l\tilde{\rho})\right\|_{2+\alpha} \leq \varepsilon\,\|\psi_l^*(\phi_l\tilde{\rho})\|_{3+\alpha} + C\|\tilde{\rho}\|_{3+\beta,\Gamma}$$

holds for $\tilde{\rho} \in h^{3+\alpha}(\Gamma)$ and $1 \leq l \leq m$.

The proof is straightforward, but lengthy. The detail can be found in Onodera [15]. Let us now complete the proof of Theorem 3.3.

Proof of Theorem 3.3. We only need to prove the resolvent estimate (3.15). For simplicity, we will denote $C > 0$ a generic constant. Combining (3.14) and Lemma 3.8 with sufficiently small $\varepsilon > 0$, we see that

$$\begin{aligned}|\lambda| \| \psi_l^* (\phi_l \tilde{\rho}) \|_{2+\alpha} + \| \psi_l^* (\phi_l \tilde{\rho}) \|_{3+\alpha} &\le C \| (\lambda I - \mathcal{W}_0^{(l)}) \psi_l^* (\phi_l \tilde{\rho}) \|_{2+\alpha} \\ &\le C \left(\| \psi_l^* (\phi_l (\lambda I - W) \tilde{\rho}) \|_{2+\alpha} + \| \tilde{\rho} \|_{3+\beta,\Gamma} \right)\end{aligned}$$

holds for any $\tilde{\rho} \in h^{3+\alpha}(\Gamma)$, $\lambda \in \{ z \in \mathbb{C} \mid \operatorname{Re} z \ge \lambda_0 \}$, and $1 \le l \le m$. Since

$$\tilde{\rho} \mapsto \max_{1 \le l \le m} \| \psi_l^* (\phi_l \tilde{\rho}) \|_{k+\alpha}$$

defines an equivalent norm on $h^{k+\alpha}(\Gamma)$ $(k = 2, 3)$, the above inequality implies

$$|\lambda| \| \tilde{\rho} \|_{2+\alpha,\Gamma} + \| \tilde{\rho} \|_{3+\alpha,\Gamma} \le C \left(\| (\lambda I - W) \tilde{\rho} \|_{2+\alpha,\Gamma} + \| \tilde{\rho} \|_{3+\beta,\Gamma} \right).$$

Then, using the interpolation inequality

$$\| \tilde{\rho} \|_{3+\beta,\Gamma} \le \varepsilon \| \tilde{\rho} \|_{3+\alpha,\Gamma} + C \| \tilde{\rho} \|_{2+\alpha,\Gamma},$$

we deduce that

$$|\lambda| \| \tilde{\rho} \|_{2+\alpha,\Gamma} + \| \tilde{\rho} \|_{3+\alpha,\Gamma} \le C \| (\lambda I - W) \tilde{\rho} \|_{2+\alpha,\Gamma}$$

holds for any $\tilde{\rho} \in h^{3+\alpha}(\Gamma)$ and $\lambda \in \{ z \in \mathbb{C} \mid \operatorname{Re} z \ge \lambda_* \}$ with sufficiently large $\lambda_* > \lambda_0$. This is nothing but (3.15). □

Theorem 1.4 now follows from Theorem 3.3 and the theory of maximal regularity of Da Prato and Grisvard [5], since $h^{2+\alpha}(\Gamma)$ is characterized as a continuous interpolation space between $h^{3+\alpha'}(\Gamma)$ and $h^{2+\alpha'}(\Gamma)$ with $0 < \alpha' < \alpha < 1$. For the proof of the solvability of fully-nonlinear equations in continuous interpolation spaces, we refer to Angenent [3, Theorem 2.7] and Lunardi [13].

4 Bifurcation criterion for quadrature surfaces

Theorems 1.2 and 1.4 immediately deduce Corollary 1.5.

Proof of Corollary 1.5. Assuming the existence of a curve $s \mapsto (\Gamma(s), t(s))$, let us derive a contradiction. We divide the proof into two cases: (i) $t'(0) > 0$ and (ii) $t'(0) = 0$.

In the case (i), we can take the inverse function t^{-1} of $t = t(s)$ at least in a neighborhood of $s = 0$. Setting

$$\tilde{\Gamma}(\tau) := \Gamma(t^{-1}(\tau)),$$

we see that $\{\tilde{\Gamma}(\tau)\}_{0\le\tau<\tilde{\varepsilon}}$ with small $\tilde{\varepsilon}$ is an $h^{3+\alpha}$ family of surfaces satisfying

$$\int_{\partial\Omega(0)} h\,d\mathcal{H}^{N-1} + \tau\int h\,d\mu = \int_{\tilde{\Gamma}(\tau)} h\,d\mathcal{H}^{N-1}$$

for harmonic functions h. Then, it follows from Theorem 1.2 that $\{\tilde{\Gamma}(\tau)\}_{0\le\tau<\tilde{\varepsilon}}$ is a solution to (1.5). However, the uniqueness assertion in Theorem 1.4 implies that $\tilde{\Gamma}(\tau) = \partial\Omega(\tau)$, or $\Gamma(s) = \partial\Omega(t(s))$. This is a contradiction.

In the case (ii), by differentiating the identity

$$\int_{\partial\Omega(0)} h\,d\mathcal{H}^{N-1} + t(s)\int h\,d\mu = \int_{\Gamma(s)} h\,d\mathcal{H}^{N-1}$$

with respect to s at $s=0$, we have a nonzero function $v_n \in h^{2+\alpha}(\partial\Omega(0))$ satisfying

$$0 = \int_{\partial\Omega(0)} \left\{\frac{\partial h}{\partial n} + (N-1)hH\right\} v_n\,d\mathcal{H}^{N-1}$$

for all harmonic functions h defined in a neighborhood of $\overline{\Omega(0)}$. Therefore, by an argument similar to the last part of the proof of Theorem 1.2, we deduce that $v_n = 0$ on $\partial\Omega(0)$, which is again a contradiction. □

References

[1] Alt, H. W.; Caffarelli, L. A., Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325** (1981), 105–144.

[2] Amann, H., Linear and quasilinear parabolic problems. Vol. I. Abstract linear theory. Monographs in Mathematics, 89. *Birkhäuser Boston, Inc., Boston, MA*, 1995.

[3] Angenent, S. B., Nonlinear analytic semiflows. *Proc. Roy. Soc. Edinburgh Sect. A* **115** (1990), 91–107.

[4] Beurling, A., On free-boundary problems for the Laplace equation. *Sem. on Analytic Funcitons* **1**, Inst. for Advanced Study Princeton (1957), 248–263.

[5] Da Prato, G.; Grisvard, P., Equations d'évolution abstraites non linéaires de type parabolique. *Ann. Mat. Pura Appl. (4)* **120** (1979), 329–396.

[6] Escher, J.; Simonett, G., Maximal regularity for a free boundary problem. *NoDEA* **2** (1995), 463–510.

[7] Escher, J.; Simonett, G., Classical solutions for Hele-Shaw models with surface tension. *Adv. Differential Equations* **2** (1997), no. 4, 619–642.

[8] Escher, J.; Simonett, G., Classical solutions of multidimensional Hele-Shaw models. *SIAM J. Math. Anal.* **28** (1997), no. 5, 1028–1047.

[9] Gilbarg, D.; Trudinger, N. S., Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. *Springer-Verlag, Berlin*, 2001.

[10] Gustafsson, B., Applications of variational inequalities to a moving boundary problem for Hele-Shaw flows. *SIAM J. Math. Anal.*, **16** (1985), no. 2, 279–300.

[11] Gustafsson, B.; Shahgholian, H., Existence and geometric properties of solutions of a free boundary problem in potential theory. *J. Reine Angew. Math.* **473** (1996), 137–179.

[12] Henrot, A., Subsolutions and supersolutions in free boundary problems. *Ark. Mat.* **32** (1994), 79–98.

[13] Lunardi, A., Analytic semigroups and optimal regularity in parabolic problems. Progress in Nonlinear Differential Equations and their Applications, 16. *Birkhäuser Verlag, Basel*, 1995.

[14] Ni, Wei-Ming; Takagi, Izumi, On the shape of least-energy solutions to a semilinear Neumann problem. Comm. Pure Appl. Math. **44** (1991), 819–851.

[15] Onodera, M., Geometric flows for quadrature identities. *preprint.*

[16] Richardson, S., Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel. *J. Fluid Mech.* **56** (1972), 609–618.

[17] Sakai, M., Quadrature Domains. Lecture Notes in Mathematics, 934. *Springer-Verlag, Berlin-New York*, 1982.

[18] Sakai, M., Application of variational inequalities to the existence theorem on quadrature domains. *Trans. Amer. Math. Soc.* **276** (1983), 267–279.